Relationships among the

UNITARY BASES

of

WEYL, SCHWINGER, WERNER & OPPENHEIM

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Introduction. Near the conclusion of our get-acquainted lunch (20 January 2012) I asked Maximilian Schlosshauer—author of *Decoherence & the Quantum-to-Classical Transition* (2007)—what he was working on "now that decoherence is more or less behind you." "Quantum information" was his response: "What can be said about how quantum information is redistributed when two systems interact." Upon my return to my office I consulted Google, was alerted to the fact that quite a number of research centers devoted to that broad topic have in recent years sprung up around the world. One of those is the Centre for Quantum Information & Foundations,[1] which exists within the Department of Applied Mathematics & Theoretical Physics (DAMTP) at Cambridge University, where I was linked to a website that lists the publications of one Jonathan Oppenhiem,[2] a prolific young Royal Society Research Fellow. The paper that happened to catch my eye was J. Oppenheim & B. Reznik, "A probabilistic and information theoretic interpretation of quantum evolutions."[3]

I read with initial disbelief the text which led to their equation (2): "It is known that for any N there exists a [set of N^2 trace-wise orthogonal unitary matrices U_α such that any unitary U can be developed

$$\mathsf{U} = \sum_{\alpha=1}^{N^2} c_\alpha \mathsf{U}_\alpha$$

with complex amplitudes given by $c_\alpha = \frac{1}{N}\mathrm{tr}(\mathsf{U}_\alpha^+ \,\mathsf{U}_\alpha)$]." I was aware that, given any linearly independent N^2-member set of *hermitian* matrices (or, indeed, of any $N \times N$ matrices, whether hermitian or not) one can, by a Gram-Schmidt orthogonalization procedure,[4] construct trace-wise orthogonal *hermitian*

[1] http://www.qi.damtp.cam.ac.uk/

[2] http://www.damtp.cam.ac.uk/user/jono/

[3] Phys. Rev. **A71**, 022312 (2004), on line at arXiv:quant-ph/0309110.

[4] For details see my "Trace-wise Orthogonal Matrices 1" (a *Mathematica* notebook dated 1 February 2012).

matrices H_α that permit one to write

$$\mathsf{H} = \sum_{\alpha=1}^{N^2} c_\alpha \mathsf{H}_\alpha \quad \text{with} \quad c_\alpha = \text{tr}(\mathsf{H}_\alpha \mathsf{H})$$

But while the space of hermitian matrices is *additively* closed, the space of unitary matrices is *multiplicatively* closed; real linear combinations of hermitian matrices are invariably hermitian, but sums of unitary matrices are typically not unitary. Whence the scepticism with which I read the casual claim on which Oppenheim and Reznik base their paper. My scepticism was misplaced. For—as belatedly occurred to me—I had encountered and made essential use of tracewise-unitary bases already nearly sixty years ago.

Oppenheim cites a paper by Reinhard Werner[5], where Werner borrows what he calls "the best known construction for unitary bases" from one of his own recent papers,[6] a construction that makes elegantly effective use of both (complex) Hadamard matrices and Latin squares. But in his Appendix Oppenheim sketches a more transparent alternative to Werner's construction, which he attributes to Schwinger.[7] The Schwinger paper—one of a set of three papers based upon quantum lectures he had been presenting at Harvard since 1951[8]—acknowledges descent from material that can be found in Chapter 4, §14 ("Quantum kinematics as an Abelian group of rotations) of Hermann Weyl's *The Theory of Groups & Quantum Mechanics* (1930). That passage in Weyl's classic (which in my own copy bears a tattered page marker) marks the first appearance of the "Weyl correspondence," which lies at the foundation of the "phase space formulation of quantum mechanics"—a subject to which, as it happens, Werner himself has made substantial contributions.[9]

Weyl's unitary basis. At page 4 in Chapter 2 of my *Advanced Quantum Topics* (2000) I introduce manifestly hermitian operators

$$\mathsf{E}(\alpha,\beta) = e^{\frac{i}{\hbar}(\alpha\,\mathsf{p}+\beta\,\mathsf{x})}$$

and establish (by appeal to basic Campbell-Baker-Hausdorff theory and some rudimentary Fourier analysis) that

$$\tfrac{1}{h}\,\text{tr}\,\mathsf{E}(\alpha,\beta) = \delta(\alpha)\delta(\beta)$$

[5] "All teleportation and dense coding schemes," arXiv:quant-ph/0003070v1, 17 Mar 2000. See especially §4 "Constructing bases of unitaries."

[6] K. G. H. Vollbrecht & R. F. Werner, "Why two qubits are special," arXiv: quant-ph/9910064v1 (14 Oct 1999).

[7] J. Schwinger, "Unitary operator bases," PNAS **46**, 570 (1960).

[8] The other papers in the series are "The algebra of microscopic measurement," PNAS **45**, 1542 (1959) and "The geometry of quantum states," PNAS **46**, 257 (1960). All are reproduced in Schwinger's *Quantum Kinematics & Dynamics* (1970). I acquired and gave close attention to Schwinger's unpublished class notes in ∼1958.

[9] See the papers listed at http://www.itp.uni-hanover.de/∼werner/Werner ByTopic.html#j6, which appeared between 1984 and 2004.

from which it is shown to follow that the operators $\mathsf{E}(\alpha,\beta)$ and $\mathsf{E}(\alpha',\beta')$ are *trace-wise orthogonal* in the sense that

$$\tfrac{1}{h}\,\mathrm{tr}\{\mathsf{E}(\alpha',\beta')\mathsf{E}^+(\alpha,\beta)\} = \delta(\alpha'-\alpha)\delta(\beta'-\beta)$$

One is led thus to an *operator analog of the Fourier integral theorem*

$$\mathsf{A} = \iint \big\{\tfrac{1}{h}\,\mathrm{tr}\big[\mathsf{A}\,\mathsf{E}^+(\alpha,\beta)\big]\big\}\mathsf{E}(\alpha,\beta)\,d\alpha d\beta \quad : \quad \text{all } \mathsf{A}$$

The details of how, by means of this fact, one gets from $\mathsf{A} = |\psi)(\psi|$ to the Wigner distribution function $P_\psi(x,p)$ are interesting, but of no immediate relevance.

Pauli's unitary basis. The Pauli matrices

$$\boldsymbol{\sigma}_0 = \mathsf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \boldsymbol{\sigma}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \boldsymbol{\sigma}_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \boldsymbol{\sigma}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

are linearly independent, so span the vector space of real/complex 2×2 matrices. They derive their special importance and utility from three circumstances:

- each of the $\boldsymbol{\sigma}$-matrices is **hermitian**;
- each of the $\boldsymbol{\sigma}$-matrices is **unitary**;
- the $\boldsymbol{\sigma}$-matrices are **trace-wise orthonormal**:

$$\tfrac{1}{2}\mathrm{tr}\big(\boldsymbol{\sigma}_i\boldsymbol{\sigma}_j\big) = \delta_{ij}$$

From the latter circumstance it follows moreover that

- each of the $\boldsymbol{\sigma}$-matrices (with the exception only of $\boldsymbol{\sigma}_0$) is **traceless**;

Every 2×2 matrix can be developed

$$\mathsf{A} = \sum_{k=0}^{3} a_k\boldsymbol{\sigma}_k \quad \text{with} \quad a_k = \tfrac{1}{2}\mathrm{tr}\big(\mathsf{A}\,\boldsymbol{\sigma}_k\big)$$

In particular, we have

$$\boldsymbol{\sigma}_i\boldsymbol{\sigma}_j = \sum_{k=0}^{3} c_{ijk}\boldsymbol{\sigma}_k \quad \text{with} \quad c_{ijk} = \tfrac{1}{2}\mathrm{tr}\big(\boldsymbol{\sigma}_i\boldsymbol{\sigma}_j\boldsymbol{\sigma}_k\big)$$

and might be surprised by the discovery that such sums invariably contain but a single term, were we not already familiar with these famous statements:

$$\begin{gathered} \boldsymbol{\sigma}_1^2 = \boldsymbol{\sigma}_2^2 = \boldsymbol{\sigma}_3^2 = \mathsf{I} \\ \boldsymbol{\sigma}_1\boldsymbol{\sigma}_2 = i\boldsymbol{\sigma}_3 = -\boldsymbol{\sigma}_2\boldsymbol{\sigma}_1 \\ \boldsymbol{\sigma}_2\boldsymbol{\sigma}_3 = i\boldsymbol{\sigma}_1 = -\boldsymbol{\sigma}_3\boldsymbol{\sigma}_2 \\ \boldsymbol{\sigma}_3\boldsymbol{\sigma}_1 = i\boldsymbol{\sigma}_2 = -\boldsymbol{\sigma}_1\boldsymbol{\sigma}_3 \end{gathered}$$

Equivalent to those statements are the following: if

$$\begin{aligned} \mathsf{A} &= (a_0\boldsymbol{\sigma}_0 + a_1\boldsymbol{\sigma}_1 + a_2\boldsymbol{\sigma}_2 + a_3\boldsymbol{\sigma}_3) = a_0\boldsymbol{\sigma}_0 + \boldsymbol{a}\cdot\boldsymbol{\sigma} \\ \mathsf{B} &= (b_0\boldsymbol{\sigma}_0 + b_1\boldsymbol{\sigma}_1 + b_2\boldsymbol{\sigma}_2 + b_3\boldsymbol{\sigma}_3) = b_0\boldsymbol{\sigma}_0 + \boldsymbol{b}\cdot\boldsymbol{\sigma} \end{aligned}$$

then

$$\mathsf{A}\,\mathsf{B} = (a_0b_0 + \boldsymbol{a}\cdot\boldsymbol{b})\boldsymbol{\sigma}_0 + (a_0\boldsymbol{b} + b_0\boldsymbol{a} + i\,\boldsymbol{a}\times\boldsymbol{b})\cdot\boldsymbol{\sigma}$$

which has this immediate consequence: if the “conjugate” of A is defined

$$\mathsf{A}_\mathrm{T} = a_0\boldsymbol{\sigma}_0 - \boldsymbol{a}\cdot\boldsymbol{\sigma}$$

then

$$\mathsf{A}\,\mathsf{A}_\mathrm{T} = (a_0a_0 - \boldsymbol{a}\cdot\boldsymbol{a})\,\mathsf{I}$$

gives

$$\mathsf{A}^{-1} = \frac{1}{a_0a_0 - \boldsymbol{a}\cdot\boldsymbol{a}}\,\mathsf{A}_\mathrm{T}$$

which on comparison with

$$\mathsf{A}^{-1} = \frac{\text{transposed matrix of cofactors}}{\text{determinant}}$$

provides

$$\det\mathsf{A} = a_0a_0 - \boldsymbol{a}\cdot\boldsymbol{a}$$

These results are structurally reminiscent of

$$(x+iy)^{-1} = \frac{1}{x^2+y^2}(x-iy)$$

It is immediately evident that

$$\mathsf{A}\text{ hermitian} \iff \text{Pauli coordinates } a_k \text{ are all real}$$

but the coordinate conditions that follow from and imply unitarity are not quite so obvious. The unitarity condition $\mathsf{A}^+ = \mathsf{A}^{-1}$ can be written

$$\bar{a}_0\boldsymbol{\sigma}_0 + \bar{\boldsymbol{a}}\cdot\boldsymbol{\sigma} = \frac{a_0\boldsymbol{\sigma}_0 - \boldsymbol{a}\cdot\boldsymbol{\sigma}}{a_0a_0 - \boldsymbol{a}\cdot\boldsymbol{a}}$$

Let the complex numbers a_k be written in polar form $a_k = r_ke^{i\theta_k}$ and write $a_0a_0 - \boldsymbol{a}\cdot\boldsymbol{a} = De^{2i\delta}$. We then require

$$\begin{aligned} e^{-i\theta_0} &= +D^{-1}e^{-2i\delta}e^{i\theta_0} \\ e^{-i\theta_k} &= -D^{-1}e^{-2i\delta}e^{i\theta_k} \quad : \quad k = 1,2,3 \end{aligned}$$

The first equation supplies $D = 1$ and $\delta = \theta_0$ while the k^{th} equation supplies $\theta_k = \delta + \pi$. So we have

$$\begin{aligned} a_0 &= \ r_0e^{i\delta} \\ a_k &= i\,r_ke^{i\delta} \end{aligned}$$

where

$$a_0a_0 - \boldsymbol{a}\cdot\boldsymbol{a} = (r_0^2 + r_1^2 + r_2^2 + r_3^2)\,e^{2i\delta} = e^{2i\delta}$$

forces the real 4-vector $\{r_0, r_1, r_2, r_3\}$ to be a unit vector. We conclude that 2×2 unitary matrices can, in the most general case, be described

$$\mathsf{U} = e^{i\delta}\{\cos\phi\cdot\boldsymbol{\sigma}_0 + i\sin\phi\cdot(\lambda_1\boldsymbol{\sigma}_1 + \lambda_2\boldsymbol{\sigma}_2 + \lambda_3\boldsymbol{\sigma}_3)\}$$

where $\boldsymbol{\lambda}$ is a real unit 3-vector. One has $\det \mathbf{U} = e^{2i\delta}$, so $\mathbf{U}$ becomes unimodular when $\delta \equiv 0 \bmod 2\pi$

In anticipation of things to come, we note that the Pauli matrices share the property that only one element in each row/column is non-zero: they are, in other words, what I will later call matrices of "shifted diagonal structure," of what Werner calls "of shift and multiply type." Moreover, the non-zero elements of each are (not, as will later seem more natural, square roots of unity but) 4^{th} roots of unity.

Dirac's unitary basis. In 1961 I looked closely to the "generalized Dirac algebra" $\mathcal{D}$ that arises from the anticommutation relation

$$\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2g_{\mu\nu}$$

where $g_{\mu\nu}$ is allowed to be any non-singular real symmetric matrix. That is essentially a Clifford algebra of order 4, with a total of $2^4-1 = 15$ basic elements (apart from the identity). I concentrated then on developing the equivalence of "similarity transformations within $\mathcal{D}$" and "rotations within a 6-space endowed with a certain induced metric G_{ij}," and paid no attention to the "unitary basis" aspects of the subject, of which I was then oblivious. Recently I revisited the subject[10] with those aspects specifically in mind. The remarks that follow draw substantially upon that recent discussion.

Though I will occasionally allude in passing to the form that expressions assume when the metric is unspecialized, I restrict my remarks to the Euclidean case. One verifies by calculation that a set of matrices that satisfy

$$\boldsymbol{\gamma}_i\boldsymbol{\gamma}_j + \boldsymbol{\gamma}_j\boldsymbol{\gamma}_i = 2\,\delta_{ij}\,\mathbf{I}$$

can be described

$$\begin{aligned}
\boldsymbol{\gamma}_1 &= \boldsymbol{\sigma}_3 \otimes \boldsymbol{\sigma}_0\\
\boldsymbol{\gamma}_2 &= \boldsymbol{\sigma}_2 \otimes \boldsymbol{\sigma}_1\\
\boldsymbol{\gamma}_3 &= \boldsymbol{\sigma}_2 \otimes \boldsymbol{\sigma}_2\\
\boldsymbol{\gamma}_4 &= \boldsymbol{\sigma}_2 \otimes \boldsymbol{\sigma}_3
\end{aligned}$$

which when spelled out look like this:

$$\boldsymbol{\gamma}_1 = \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \boldsymbol{\gamma}_2 = \begin{pmatrix} 0 & 0 & 0 & -i\\ 0 & 0 & -i & 0\\ 0 & i & 0 & 0\\ i & 0 & 0 & 0 \end{pmatrix}$$
$$\boldsymbol{\gamma}_3 = \begin{pmatrix} 0 & 0 & 0 & -1\\ 0 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad \boldsymbol{\gamma}_4 = \begin{pmatrix} 0 & 0 & -i & 0\\ 0 & 0 & 0 & i\\ i & 0 & 0 & 0\\ 0 & -i & 0 & 0 \end{pmatrix}$$

[10] See "Trace-wise Orthogonal Matrices 4" (*Mathematica* notebook dated 17 February 2012) and also "Aspects of the theory of Clifford algebras": notes for a seminar prresented 27 March 1968 to the Reed College Math Club.

Each of those matrices is unitary, so all products of $\boldsymbol{\gamma}$-matrices are assuredly unitary.

Working from statements that in the general theory read $\sigma_{\mu\nu} = \gamma_\mu\gamma_\nu - g_{\mu\nu}$ and by $\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2g_{\mu\nu}$ entail $\sigma_{\mu\nu} = -\sigma_{\nu\mu}$, we now introduce

$$\boldsymbol{\sigma}_{ij} = -\boldsymbol{\sigma}_{ij} = i\boldsymbol{\gamma}_i\boldsymbol{\gamma}_j$$

where the i-factors have been introduced to achieve trace-wise normality (see below). Explicitly,

$$\boldsymbol{\sigma}_{12} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \ \boldsymbol{\sigma}_{13} = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \ \boldsymbol{\sigma}_{14} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

$$\boldsymbol{\sigma}_{23} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \ \boldsymbol{\sigma}_{24} = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}$$

$$\boldsymbol{\sigma}_{34} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

Next we introduce what is sometimes called the "5^{th} Dirac matrix" and in general theory is constructed $\Gamma = \frac{1}{4!}\epsilon^{\mu\nu\rho\sigma}\sigma_{\mu\nu}\sigma_{\rho\sigma}$ but in the Euclidean case (after abandonment of a factor of $i^2 = -1$) becomes simply

$$\boldsymbol{\Gamma} = \boldsymbol{\gamma}_1\boldsymbol{\gamma}_2\boldsymbol{\gamma}_3\boldsymbol{\gamma}_4 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Finally, we introduce matrices

$$\boldsymbol{\lambda}_k = i\,\boldsymbol{\Gamma}\boldsymbol{\gamma}_k = \begin{cases} -\,i\boldsymbol{\gamma}_2\boldsymbol{\gamma}_3\boldsymbol{\gamma}_4 \\ +\,i\boldsymbol{\gamma}_1\boldsymbol{\gamma}_3\boldsymbol{\gamma}_4 \\ -\,i\boldsymbol{\gamma}_1\boldsymbol{\gamma}_2\boldsymbol{\gamma}_4 \\ +\,i\boldsymbol{\gamma}_1\boldsymbol{\gamma}_2\boldsymbol{\gamma}_3 \end{cases}$$

where the i-factors serve the same objective as before. Explicitly

$$\boldsymbol{\lambda}_1 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}, \ \boldsymbol{\lambda}_2 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\boldsymbol{\lambda}_3 = \begin{pmatrix} 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}, \ \boldsymbol{\lambda}_4 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

In previous work I have written

$$\mathsf{G} = S\mathsf{I} + \sum_{i=1}^{4} V_i\boldsymbol{\gamma}_i + \tfrac{1}{2}\sum_{i,j=1}^{4} T_{ij}\boldsymbol{\sigma}_{ij} + \sum_{j=1}^{4} A_j\boldsymbol{\lambda}_j + P\boldsymbol{\Gamma}$$

to describe the general element of the Dirac algebra $\mathcal{D}$, where the coefficients have been assigned names intended to reflect the physically significant distinctions among Scalars, Vectors, Tensors, Axial vectors and Pseudoscalars. But for present purposes I find it convenient to adopt a single-index notation

$$\begin{array}{ccccc} & & \boldsymbol{\epsilon}_5 = \boldsymbol{\sigma}_{12} & & \\ & \boldsymbol{\epsilon}_1 = \boldsymbol{\gamma}_1 & \boldsymbol{\epsilon}_6 = \boldsymbol{\sigma}_{13} & \boldsymbol{\epsilon}_{11} = \boldsymbol{\lambda}_1 & \\ & \boldsymbol{\epsilon}_2 = \boldsymbol{\gamma}_2 & \boldsymbol{\epsilon}_7 = \boldsymbol{\sigma}_{14} & \boldsymbol{\epsilon}_{12} = \boldsymbol{\lambda}_2 & \\ \boldsymbol{\epsilon}_0 = \mathsf{I} & & & & \boldsymbol{\epsilon}_{15} = \boldsymbol{\Gamma} \\ & \boldsymbol{\epsilon}_3 = \boldsymbol{\gamma}_3 & \boldsymbol{\epsilon}_8 = \boldsymbol{\sigma}_{23} & \boldsymbol{\epsilon}_{13} = \boldsymbol{\lambda}_3 & \\ & \boldsymbol{\epsilon}_4 = \boldsymbol{\gamma}_4 & \boldsymbol{\epsilon}_9 = \boldsymbol{\sigma}_{24} & \boldsymbol{\epsilon}_{14} = \boldsymbol{\lambda}_4 & \\ & & \boldsymbol{\epsilon}_{10} = \boldsymbol{\sigma}_{34} & & \end{array}$$

writing

$$\mathsf{G} = \sum_{k=0}^{15} g_k\boldsymbol{\epsilon}_k$$

With the assistance of *Mathematica* we quickly establish—in precise mimicry of the situation encountered in the Pauli algebra—that

- each of the $\boldsymbol{\epsilon}$-matrices is **hermitian**;
- each of the $\boldsymbol{\epsilon}$-matrices is **unitary**;
- the $\boldsymbol{\epsilon}$-matrices are **trace-wise orthonormal**.

From the latter circumstance it follows moreover that

- each of the $\boldsymbol{\epsilon}$-matrices (with the exception only of $\boldsymbol{\epsilon}_0$) is **traceless**;

The $\boldsymbol{\epsilon}$-matrices are linearly independent, and equal in number to the number of elements in a 4×4 matrix, so span the space of such matrices, in which they constitute a trace-wise orthonormal hermitian basis which is also a unitary basis. Every such matrix can be developed

$$\mathsf{A} = \sum_{k=0}^{15} a_k\boldsymbol{\epsilon}_k \quad \text{with} \quad a_k = \tfrac{1}{4}\mathrm{tr}\big(\mathsf{A}\,\boldsymbol{\epsilon}_k\big)$$

In particular, we have

$$\boldsymbol{\epsilon}_i\boldsymbol{\epsilon}_j = \sum_{k=0}^{15} c_{ijk}\boldsymbol{\epsilon}_k \quad \text{with} \quad c_{ijk} = \tfrac{1}{4}\mathrm{tr}\big(\boldsymbol{\epsilon}_i\boldsymbol{\epsilon}_j\boldsymbol{\epsilon}_k\big)$$

From (i) the fact that

$$\text{every } \boldsymbol{\epsilon} \text{ is of the form } \boldsymbol{\gamma}_1^{\alpha_1}\boldsymbol{\gamma}_2^{\alpha_2}\boldsymbol{\gamma}_3^{\alpha_3}\boldsymbol{\gamma}_4^{\alpha_4} \text{ with all } \alpha\in\{0,1\}$$

and (ii) the fact that $\boldsymbol{\gamma}_i\boldsymbol{\gamma}_j + \boldsymbol{\gamma}_j\boldsymbol{\gamma}_i = 2\delta_{ij}\mathsf{I}$ can be used to simplify products of $\boldsymbol{\epsilon}$-matrices (return them to single-$\boldsymbol{\epsilon}$ form) it follows readily that for given i and j only one of the c_{ijk} is non-zero.

It follows immediately from the hermiticity of the $\boldsymbol{\epsilon}$-matrices that

$$\mathsf{A} \text{ hermitian} \iff \text{Dirac coordinates } a_k \text{ are all real}$$

but the detailed significance of the coordinate conditions that follow from and imply unitarity—which can be phrased

$$\bar{a}_k = \tfrac{1}{4}\,\mathrm{tr}\big(\mathbf{A}^{-1}\boldsymbol{\epsilon}_k\big)$$

—is far from obvious. Numerical evidence (look to the Dirac coordinates $\{x_k\}$ of a randomly constructed unitary matrix $\mathbf{X}$) indicates that

$$\mathbf{A} \text{ unitary } \Longrightarrow \text{ Dirac coordinate vector has unit norm: } \sum_{k=0}^{15}\bar{a}_k a_k = 1 \tag{1}$$

but that the converse is *not* true. It is in an (alas! unsuccessful) effort to clarify the situation that I look now to the **inversion problem**, as it arises within the Dirac algebra $\mathcal{D}$.[11]

We introduce now "conjugation" operations of *two* flavors:

$$\begin{aligned}
\mathbf{A} &= a_0\boldsymbol{\epsilon}_0 + \sum_{i=1}^{4} a_i\boldsymbol{\epsilon}_i + \sum_{i=5}^{10} a_j\boldsymbol{\epsilon}_j + \sum_{i=11}^{14} a_k\boldsymbol{\epsilon}_k + a_{15}\boldsymbol{\epsilon}_{15}\\
\mathbf{A}_{\mathrm{T}} &= a_0\boldsymbol{\epsilon}_0 - \sum_{i=1}^{4} a_i\boldsymbol{\epsilon}_i - \sum_{i=5}^{10} a_j\boldsymbol{\epsilon}_j + \sum_{i=11}^{14} a_k\boldsymbol{\epsilon}_k + a_{15}\boldsymbol{\epsilon}_{15}\\
\mathbf{A}_{\mathrm{t}} &= a_0\boldsymbol{\epsilon}_0 + \sum_{i=1}^{4} a_i\boldsymbol{\epsilon}_i + \sum_{i=5}^{10} a_j\boldsymbol{\epsilon}_j - \sum_{i=11}^{14} a_k\boldsymbol{\epsilon}_k - a_{15}\boldsymbol{\epsilon}_{15}
\end{aligned}$$

of which the first involves negation of the $\boldsymbol{\gamma}$ and $\boldsymbol{\sigma}$ terms, the second involves negation of the $\boldsymbol{\lambda}$ and $\boldsymbol{\Gamma}$ terms. Looking with *Mathematica*'s assistance to the development of $\mathbf{A}\mathbf{A}_{\mathrm{T}}$ we find that the coordinates

$$\tfrac{1}{4}\,\mathrm{tr}\big(\mathbf{A}\mathbf{A}_{\mathrm{T}}\boldsymbol{\epsilon}_k\big) = 0 \quad : \quad k = 1,\ldots,10$$

Looking next to

$$\mathbf{A}\mathbf{A}_{\mathrm{T}}(\mathbf{A}\mathbf{A}_{\mathrm{T}})_{\mathrm{t}} = \sum_{k=0}^{15}\tfrac{1}{4}\,\mathrm{tr}\big\{\mathbf{A}\mathbf{A}_{\mathrm{T}}(\mathbf{A}\mathbf{A}_{\mathrm{T}})_{\mathrm{t}}\boldsymbol{\epsilon}_k\big\}\boldsymbol{\epsilon}_k \equiv \sum_{k=0}^{15} c_k\boldsymbol{\epsilon}_k$$

we find that

$$c_k = 0 \quad : \quad k \neq 0$$

giving

$$\mathbf{A}\mathbf{A}_{\mathrm{T}}(\mathbf{A}\mathbf{A}_{\mathrm{T}})_{\mathrm{t}} = c_0\,\mathbf{I} \quad \Longrightarrow \quad \mathbf{A}^{-1} = \frac{\mathbf{A}_{\mathrm{T}}(\mathbf{A}\mathbf{A}_{\mathrm{T}})_{\mathrm{t}}}{c_0}$$

Evidently the matrix in the numerator (which is cubic in the coordinates of $\mathbf{A}$) provides a *factored description of the transposed matrix of cofactors*, while the

[11] Here I borrow from my 1961 notes, where I worked very carefully/patiently on large sheets of paper, in $\{S, V, T, A, P\}$ notation, which for this purpose offers tensor-theoretic advantages.

denominator (which is quartic in the coordinates of $\mathbf{A}$) provides

$$c_0 = \det \mathbf{A}$$

Detailed $\{S, V, T, V, P\}$-descriptions of numerator & denominator are developed in the 1961 notes, but are too complicated to merit transcription here.[12] The unitarity condition can now be written

$$\bar{a}_k = \tfrac{1}{4}\,\mathrm{tr}\Big\{\frac{\mathbf{A}_{\mathrm{T}}(\mathbf{A}\mathbf{A}_{\mathrm{T}})_{\mathrm{t}}}{c_0}\,\boldsymbol{\epsilon}_k\Big\}$$

which is, however, not very informative. In particular, it provides no insight into the origin of (1). We do, however, find computationally that if we take $\mathbf{A}$ to be a randomly constructed numerical unitary then the preceding equation is correct, and moreover that

$$c_0 = e^{i(\text{phase angle})}$$

We note finally that every $\boldsymbol{\epsilon}$-matrix is of "shifted diagonal form"

$$\boldsymbol{\epsilon}_k = (\text{permutation matrix})_k \cdot (\text{diagonal matrix})_k$$

and that the elements of the diagonal factor are in every instance (once again, as in the Pauli algebra) 4^{th} roots of unity.

Recapitulation, and a motivational look ahead. The Pauli and Dirac algebras are Clifford algebras of orders 2 and 4. From the generators $\{\epsilon_1, \epsilon_2, \ldots, \epsilon_n\}$ of the Clifford algebra $\mathcal{C}_n$—because they are required to satisfy the anticommutation relation $\epsilon_i\epsilon_j + \epsilon_j\epsilon_i = 2\delta_{ij}$—one can construct only $N = 2^n$ essentially distinct products, which we may take to have the form $\epsilon_1^{\alpha_1}\epsilon_2^{\alpha_2}\cdots\epsilon_n^{\alpha_n} : \alpha \in \{0, 1\}$. When n is even ($n = 2\nu$) we have $N = 2^\nu \times 2^\nu$, which is the number of elements in a $2^\nu \times 2^\nu$-dimensional matrix. In such cases we may expect to be able to construct $2^\nu \times 2^\nu$-dimensional matrix representations $\{\boldsymbol{\epsilon}_0, \boldsymbol{\epsilon}_2, \ldots, \boldsymbol{\epsilon}_{N-1}\}$ of the base elements of $\mathcal{C}_n$. We might expect, moreover, to be able to construct those matrices from ν-fold Kronecker products of Pauli matrices, and therefore to be in position to extract their properties relatively painlessly from those of the Pauli matrices. Specifically, we might expect to be able to arrange for the elements of $\{\boldsymbol{\epsilon}_0, \boldsymbol{\epsilon}_2, \ldots, \boldsymbol{\epsilon}_{N-1}\}$ to be hermitian, unitary, and trace-wise orthonormal. Google reports the existence of a sizeable literature pertaining to "higher-dimensional Dirac matrices."[13] I do not pursue this subject because it would appear to hold promise of providing unitary bases applicable only to matrices whose dimension is a power of 2. The unitary bases constructed by Schwinger, Werner and Oppenheim—to which I now turn—suffer from no such limitation.

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[12] They are spelled out in "Transformational principles latent in the theory of Clifford algebras" (October, 2003).

[13] See, for example, http://en.wikipedia.org/wiki/Weyl-Brauer_matrices.

Schwinger's unitary basis. Unitary operators on $\mathcal{V}_n$ send orthonormal frames to orthonormal frames. Let $\{|a_k)\}$ and $\{|b_k)\}$ be two such frames in $\mathcal{V}_n$, and define

$$\mathsf{U}_{ab} = \sum_k |a_k)(b_k|$$

Then

$$\mathsf{U}_{ab}|b_i) = |a_i) \quad : \quad \text{all } i$$

Obviously $\mathsf{U}_{ab}^{+} = \mathsf{U}_{ba}$, $\mathsf{U}_{ab}\mathsf{U}_{bc} = \mathsf{U}_{ac}$, $\mathsf{U}_{ab}\mathsf{A}\,\mathsf{U}_{ba} = (\text{tr}\,\mathsf{A})\mathsf{I}$. Julian Schwinger, in "Unitary operator bases," PNAS **46**, 570 (1960), discusses various aspects of such operators, drawing motivation it would appear mainly from previous papers in the same series.[14] I look here only to the material of immediate relevance.

Let the elements of the "home frame" in $\mathcal{V}_n$ be denoted

$$|0) = \begin{pmatrix} 1\\0\\0\\\vdots\\0 \end{pmatrix}, \quad |1) = \begin{pmatrix} 0\\1\\0\\\vdots\\0 \end{pmatrix}, \quad \ldots, \quad |n-1) = \begin{pmatrix} 0\\0\\0\\\vdots\\1 \end{pmatrix}$$

where the indices run $0, 1, \ldots, n-1$ in order to facilitate the modular arithmetic that will soon come into play. Schwinger looks to the unitary matrix

$$\mathsf{P} = \begin{pmatrix} 0 & 1 & 0 & 0 & \ldots & 0 & 0\\ 0 & 0 & 1 & 0 & \ldots & 0 & 0\\ 0 & 0 & 0 & 1 & \ldots & 0 & 0\\ 0 & 0 & 0 & 0 & \ldots & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & 0 & \ldots & 0 & 1\\ 1 & 0 & 0 & 0 & \ldots & 0 & 0 \end{pmatrix}$$

that sends the home frame into the frame in which the indices have been cyclically incremented:

$$\mathsf{P}|k) = |k-1 \text{ mod } n)$$

P is a permutation matrix, clearly unitary in view of its frame $\longrightarrow$ frame action. It is clear also that

$$\mathsf{P}^n = \mathsf{I}$$

[14] The others are "The algebra of microscopic measurement," PNAS **45**, 1542 (1959) and "The geometry of quantum states," PNAS **46**, 257 (1960). Those papers provide the first published account of material that Schwinger had been presenting in his quantum mechanics classes at Harvard, beginning in about 1951. The material was presented at the Les Houches Summer School in 1955, and is reproduced in Chapters I and II of his *Quantum Kinematics and Dynamics* (1970).

so by the "backward Hamilton-Jacobi theorem" the characteristic equation of P reads $\omega^n = 1$. The eigenvalues of P are therefore n^{th} roots of unity:

$$\{\omega^0, \omega^1, \omega^2, \dots, \omega^{n-1}\} \quad \text{with} \quad \omega = e^{i\,2\pi/n}$$

The characteristic equation of an $n \times n$ matrix X can in the general case be written

$$x^n - x^{n-1}\,\text{tr}\,\mathsf{X} + \cdots + (-1)^n\,\det\mathsf{X} = 0$$

which in the present instance informs us that

$$\mathsf{P}\text{ is traceless: }\ \text{tr}\,\mathsf{P} = 0$$

(which is anyway obvious) and that $\det\mathsf{P} = -(-1)^n$:[15]

The rotation matrix P is proper/improper according as n is odd/even

The normalized eigenvectors of P are now immediately evident: they are

$$|u_0) = \tfrac{1}{\sqrt{n}}\begin{pmatrix}1\\1\\1\\\vdots\\1\end{pmatrix},\ |u_1) = \tfrac{1}{\sqrt{n}}\begin{pmatrix}1\\\omega\\\omega^2\\\vdots\\\omega^{n-1}\end{pmatrix},\ |u_2) = \tfrac{1}{\sqrt{n}}\begin{pmatrix}1\\\omega^2\\\omega^4\\\vdots\\\omega^{(n-1)2}\end{pmatrix},\ \dots$$

$$|u_k) = \tfrac{1}{\sqrt{n}}\begin{pmatrix}1\\\omega^k\\\omega^{2k}\\\omega^{3k}\\\vdots\\\omega^{(n-1)k}\end{pmatrix}$$

where all exponents are to be read "mod n". In short,

$$|u_k) = \tfrac{1}{\sqrt{n}}\sum_{j=0}^{n-1}\omega^{jk\,\text{mod}\,n}|j)$$

Schwinger directs our attention next to the unitary matrix Q that achieves (reversed) cyclic permutation within the $\{|u)\}$-frame:

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[15] This follows also from

$$\det\mathsf{P} = \text{product of eigenvalues} = e^{i(n-1)\pi}$$

$$\mathsf{Q} = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & \omega & 0 & 0 & \dots & 0 \\ 0 & 0 & \omega^2 & 0 & \dots & 0 \\ 0 & 0 & 0 & \omega^3 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \omega^{n-1} \end{pmatrix} \quad : \quad \mathsf{Q}|u_k) = |u_{k+1 \bmod n})$$

The unitarity of Q is obvious (obvious also from its defining action). Again,

$$\mathsf{Q}^n = \mathsf{I}$$

so Q has the same spectum as P (read the diagonal!) and possesses the same trace and determinant:

$$\begin{aligned} \operatorname{tr}\mathsf{Q} &= \text{sum of eigenvalues} = 0 \\ \det\mathsf{Q} &= \text{product of eigenvalues} = e^{i(n-1)\pi} = (-1)^{n-1} \end{aligned}$$

All products of P and Q matrices are unitary, in which connection it is important to notice that P and Q fail to commute: by quick calculation

$$\mathsf{P}\,\mathsf{Q} = \omega\,\mathsf{Q}\,\mathsf{P} \tag{2.1}$$

from which it follows that

$$\mathsf{P}^\mu\mathsf{Q}^\nu = \omega^{\mu\nu \bmod n}\,\mathsf{Q}^\nu\mathsf{P}^\mu \tag{2.2}$$

Schwinger remarks in passing that

$$\mathsf{P}^\mu\mathsf{Q}^\nu = \omega^{\mu\nu \bmod n}\,\mathsf{Q}^\nu\mathsf{P}^\mu \quad\iff\quad \mathsf{Q}^{-\mu}\mathsf{P}^\nu = \omega^{\mu\nu \bmod n}\,\mathsf{P}^\nu\mathsf{Q}^{-\mu}$$

establishes the sense in which P and Q are "complimentary": all statements deduced from (2.1) remain valid under the substitutional transformation

$$\mathsf{P} \to \mathsf{Q} \qquad\qquad \mathsf{Q} \to \mathsf{P}^{-1}$$

An equation of precisely the form (2.1) appears at the beginning of and is central to Weyl's discussion of "Quantum Kinematics."[16]

Following Schwinger, we look now to the n^2-member population of $n \times n$ unitary matrices

$$\mathsf{U}_{\mu\nu} = \mathsf{P}^\mu\mathsf{Q}^\nu \quad : \quad \mu,\nu \in \{0,1,\dots,n-1\}$$

which we will show to be trace-wise orthonormal:

$$\tfrac{1}{n}\operatorname{tr}(\mathsf{U}^+_{\mu\nu}\,\mathsf{U}_{\rho\sigma}) = \delta_{\mu\rho}\delta_{\nu\sigma} \equiv \delta_{\mu\nu,\rho\sigma}$$

To that end, we insert

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[16] See again the passage cited previously in his *The Theory of Groups and Quantum Mechanics* (1930). Recall also from Campbell-Baker-Hausdorff theory (See Chapter 0, page 31 in my *Advanced Quantum Topics* (2000)) that if operators A and B commute with their commutator then

$$e^{\mathsf{A}}e^{\mathsf{B}} = \omega\; e^{\mathsf{B}}e^{\mathsf{A}} \quad \text{with} \quad \omega = e^{\frac{1}{2}[\mathsf{A},\mathsf{B}]}$$

This and a couple of closely related identites play critically important roles in the development and applications of the Weyl Correspondence.

$$\begin{aligned}\mathsf{P}^{\mu} &= \sum_{i=0}^{n-1} |i)(i+\mu \bmod n| \\ \mathsf{Q}^{\nu} &= \sum_{j=0}^{n-1} \omega^{\nu j}|j)(j|\end{aligned}$$

into

$$\tfrac{1}{n}\mathrm{tr}(\mathsf{U}^{+}_{\mu\nu}\,\mathsf{U}_{\rho\sigma}) = \tfrac{1}{n}\sum_{r=0}^{n-1}(r|\,\mathsf{Q}^{-\nu}\,\mathsf{P}^{\rho-\mu}\,\mathsf{Q}^{\sigma}|r)$$

to obtain

$$\begin{aligned}&= \tfrac{1}{n}\sum_{r=0}^{n-1}\sum_{i,j,k,l=0}^{n-1}(r|j)(j|i)(i-\mu \bmod n|k) \\ &\qquad\qquad\qquad (k+\rho \bmod n|l)(l|r)\,\omega^{-\nu j+\sigma l} \\ &= \tfrac{1}{n}\sum_{r=0}^{n-1}\sum_{k=0}^{n-1}(r-\mu \bmod n|k)(k+\rho \bmod n|r)\,\omega^{r(\sigma-\nu)} \\ &= \tfrac{1}{n}\sum_{r=0}^{n-1}\sum_{k=0}^{n-1}(r-\mu \bmod n|k)(k|r-\rho \bmod n)\,\omega^{r(\sigma-\nu)} \\ &= \tfrac{1}{n}\sum_{r=0}^{n-1}\delta_{(r-\rho \bmod n)(r-\mu \bmod n)}\,\omega^{r(\sigma-\nu)} \\ &= \delta_{\mu\rho}\cdot\tfrac{1}{n}\sum_{r=0}^{n-1}\omega^{r(\sigma-\nu)} \\ &= \delta_{\mu\rho}\delta_{\nu\sigma} \equiv \delta_{\mu\rho,\nu\sigma}\end{aligned}$$

It follows in particular, by $\mathsf{U}_{00} = \mathsf{P}^0\,\mathsf{Q}^0 = \mathsf{I}$, that

$$\tfrac{1}{n}\mathrm{tr}\,\mathsf{U}_{\rho\sigma} = \delta_{0\rho}\delta_{0\sigma}$$

so all U-matrices are traceless, with the sole exception of $\mathsf{U}_{00} = \mathsf{I}$. They are, moreover, *complete* in the sense that

$$\sum_{\mu,\nu=0}^{n-1}\mathsf{U}^{+}_{\mu\nu}\,\mathsf{A}\,\mathsf{U}_{\mu\nu} = (n\,\mathrm{tr}\,\mathsf{A})\,\mathsf{I} \quad : \quad \text{all } \mathsf{A}$$

as I demonstrate:

$$\begin{aligned}
&= \sum_{\mu,\nu=0}^{n-1} \sum_{i,j,k,l=0}^{n-1} |i)(i|j+\mu \bmod n)(j|\mathsf{A}|k)(k+\mu \bmod n|l)(l|\,\omega^{\nu(l-i)} \\
&= n\sum_{\mu=0}^{n-1} \sum_{i,j,k,l=0}^{n-1} |i)(i|j+\mu \bmod n)(j|\mathsf{A}|k)(k+\mu \bmod n|l)(l|\,\delta_{il} \\
&= n\sum_{\mu=0}^{n-1} \sum_{i,j,k=0}^{n-1} |i)(i|j+\mu \bmod n)(j|\mathsf{A}|k)(k+\mu \bmod n|i)(i| \\
&= n\sum_{i,j,k=0}^{n-1} \delta_{jk}|i)(j|\mathsf{A}|k)(i| = (n\,\mathrm{tr}\,\mathsf{A})\,\mathsf{I}
\end{aligned}$$

In the case $\mathsf{A} = \mathsf{I}$ we recover the trivial statement

$$\sum_{\mu,\nu=0}^{n-1} \mathsf{U}^{+}_{\mu\nu}\mathsf{U}_{\mu\nu} = \sum_{\mu,\nu=0}^{n-1} \mathsf{I} = n^2\,\mathsf{I}$$

But though unitary, orthonormal, traceless and complete, the Schwinger matrices $\mathsf{U}_{\mu\nu}$ are (contrast the situation in the Pauli and Dirac algebras) —with the sole exception of U_{00}—*non-hermitian*:

$$\mathsf{U}^{+}_{\mu\nu} = \mathsf{Q}^{-\nu}\,\mathsf{P}^{-\mu} = \omega^{-\mu\nu}\,\mathsf{P}^{-\mu}\,\mathsf{Q}^{-\nu} = \omega^{-\mu\nu}\,\mathsf{U}_{(n-\mu)(n-\nu)}$$

Finally, we by $\mathsf{U}_{\mu\nu}\mathsf{U}_{\rho\sigma} = \mathsf{P}^{\mu}\mathsf{Q}^{\nu}\mathsf{P}^{\rho}\mathsf{Q}^{\sigma} = \omega^{-\nu\rho}\,\mathsf{P}^{\mu+\rho}\,\mathsf{Q}^{\nu+\sigma}$ have the composition rule

$$\begin{aligned}
\mathsf{U}_{\mu\nu}\mathsf{U}_{\rho\sigma} &= \sum_{\kappa,\lambda=0}^{n-1} c_{\mu\nu\rho\sigma\kappa\lambda}\,\mathsf{U}_{\kappa\lambda} \\
&\qquad c_{\mu\nu\rho\sigma\kappa\lambda} = \delta_{\kappa,(\mu+\rho \bmod n)}\delta_{\lambda,(\nu+\sigma \bmod n)}\,\omega^{-(\nu\rho \bmod n)}
\end{aligned}$$

where again only a single term actually contributes to the $\sum_{\kappa,\lambda}$. A similar argument supplies

$$\begin{aligned}
\mathsf{U}^{+}_{\mu\nu}\mathsf{U}_{\rho\sigma} = \mathsf{Q}^{-\nu}\,\mathsf{P}^{\rho-\mu}\,\mathsf{Q}^{\sigma} &= \omega^{\nu(\rho-\mu)}\,\mathsf{P}^{\rho-\mu}\,\mathsf{Q}^{\sigma-\nu} \\
&= \omega^{\nu(\rho-\mu)}\,\mathsf{U}_{(\rho-\mu \bmod n)(\sigma-\nu \bmod n)}
\end{aligned}$$

which, if we had $\frac{1}{n}\mathrm{tr}\,\mathsf{U}_{\kappa\lambda} = \delta_{0\kappa}\delta_{0\lambda}$ already at our disposal, would supply an alternative (and much simpler) proof of trace-wise orthonormality.

Recall that the Clifford algebra $\mathcal{C}_n$ springs from an n-member set of anticommutative generators $\{\boldsymbol{\epsilon}_1, \boldsymbol{\epsilon}_2, \ldots, \boldsymbol{\epsilon}_n\}$, each of which is a square root of I. The Schwinger algebra derives much of its relative simplicity from the circumstance that it springs from only *two* generators, which are n^{th} roots of I and which satisfy a similarly simple commutivity relation.

It is partly to demonstrate that the material developed above is actually much simpler than at first sight strikes the eye (and partly to pose what I call the "unitarity problem") that I look now to the specifics of some low-dimensional cases.

Schwinger's basis in the case n = 2. Here

$$\mathsf{P} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

which has eigenvalues $\{\omega^0, \omega^1\} = \{1, -1\}$ with $\omega = e^{i2\pi/2} = -1$.[17] Therefore

$$\mathsf{Q} = \begin{pmatrix} 1 & 0 \\ 0 & \omega \end{pmatrix}$$

and the Schwinger matrices become

$$\mathsf{U}_{00} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ \mathsf{U}_{01} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \ \mathsf{U}_{10} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \mathsf{U}_{11} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

All (with the exception of U_{00}) are manifestly traceless; they are readily shown to be unitary and orthonormal

$$\tfrac{1}{2}\mathrm{tr}(\mathsf{U}^{+}_{\mu\nu}\mathsf{U}_{\rho\sigma}) = \delta_{\mu\rho}\delta_{\nu\sigma}$$

and all except U_{11} are hermitian. When the latter is multiplied by i (which does no damage to tracelessness, unitarity or orthonormality) the Schwinger matrices become Pauli matrices:

$$\mathsf{U}_{00} = \boldsymbol{\sigma}_0\,, \quad \mathsf{U}_{01} = \boldsymbol{\sigma}_3\,, \quad \mathsf{U}_{10} = \boldsymbol{\sigma}_1\,, \quad i\,\mathsf{U}_{11} = \boldsymbol{\sigma}_2$$

Arbitrary real/complex 2×2 matrices A and B can be developed

$$\mathsf{A} = \sum_{\mu,\nu=0}^{1} a_{\mu\nu}\mathsf{U}_{\mu\nu} = \begin{pmatrix} a_{0,0} + a_{0,1} & a_{1,0} - a_{1,1} \\ a_{1,0} + a_{1,1} & a_{0,0} - a_{0,1} \end{pmatrix}$$

$$\mathsf{B} = \sum_{\rho,\sigma=0}^{1} b_{\rho\sigma}\mathsf{U}_{\rho\sigma} = \begin{pmatrix} b_{0,0} + b_{0,1} & b_{1,0} - b_{1,1} \\ b_{1,0} + b_{1,1} & b_{0,0} - b_{0,1} \end{pmatrix}$$

Their product $\mathsf{C} = \mathsf{A}\,\mathsf{B}$ becomes

$$\mathsf{C} = \sum_{\mu,\nu=0}^{1} c_{\mu\nu}\mathsf{U}_{\mu\nu}$$

[17] This, by the way, is the *only* case in which all of the eigenvalues of P are real, and in which therefore all the Schwinger matrices turn out to be real.

with

$$\begin{aligned} c_{00} &= \tfrac{1}{2}\mathrm{tr}(\mathbf{U}_{00}^{+}\mathbf{A}\,\mathbf{B}) = a_{00}b_{00} + a_{01}b_{01} + a_{10}b_{10} - a_{11}b_{11} \\ c_{01} &= \tfrac{1}{2}\mathrm{tr}(\mathbf{U}_{01}^{+}\mathbf{A}\,\mathbf{B}) = a_{00}b_{01} + a_{01}b_{00} + a_{10}b_{11} - a_{11}b_{10} \\ c_{10} &= \tfrac{1}{2}\mathrm{tr}(\mathbf{U}_{10}^{+}\mathbf{A}\,\mathbf{B}) = a_{00}b_{10} - a_{01}b_{11} + a_{10}b_{00} + a_{11}b_{01} \\ c_{11} &= \tfrac{1}{2}\mathrm{tr}(\mathbf{U}_{11}^{+}\mathbf{A}\,\mathbf{B}) = a_{00}b_{11} - a_{01}b_{10} + a_{10}b_{01} + a_{11}b_{00} \end{aligned}$$

where one might alternatively have appealed to the Schwinger product rule. To obtain the coordinates of $\mathbf{B} = \mathbf{A}^{-1}$ we (with *Mathematica*'s assistance) solve the system of equations $\{c_{00} = 1,\ c_{10} = c_{01} = c_{11} = 0\}$ to obtain

$$\begin{aligned} b_{00} &= +\frac{a_{00}}{a_{00}^2 - a_{10}^2 - a_{01}^2 + a_{11}^2} \\ b_{01} &= -\frac{a_{01}}{a_{00}^2 - a_{10}^2 - a_{01}^2 + a_{11}^2} \\ b_{10} &= -\frac{a_{10}}{a_{00}^2 - a_{10}^2 - a_{01}^2 + a_{11}^2} \\ b_{11} &= -\frac{a_{11}}{a_{00}^2 - a_{10}^2 - a_{01}^2 + a_{11}^2} \end{aligned}$$

where

$$a_{00}^2 - a_{10}^2 - a_{01}^2 + a_{11}^2 \equiv A = \det\mathbf{A}$$

Unitarity requires (recall that $\mathbf{U}_{11}^{+} = -\mathbf{U}_{11}$)

$$\left.\begin{aligned} b_{00} &= +\bar{a}_{00} = +a_{00}/A \\ b_{01} &= +\bar{a}_{01} = -a_{01}/A \\ b_{10} &= +\bar{a}_{10} = -a_{10}/A \\ b_{11} &= -\bar{a}_{11} = -a_{11}/A \end{aligned}\right\} \Longrightarrow A = e^{2i\alpha}$$

so we have

$$\begin{aligned} a_{00} &= \ \ r_{00}e^{i\alpha} \\ a_{01} &= i\,r_{01}e^{i\alpha} \\ a_{10} &= i\,r_{10}e^{i\alpha} \\ a_{11} &= \ \ r_{11}e^{i\alpha} \end{aligned}$$

where the numbers $\{r_{00}, r_{10}, r_{01}, r_{11}\}$ are real. We have now in hand the coordinate conditions that are necessary and sufficient for $\mathbf{A}$ to be unitary (or, for that matter, unimodular: set $\alpha = 0$), and are in position to write

$$\begin{aligned} \det\mathbf{A} = a_{00}^2 - a_{10}^2 - a_{01}^2 + a_{11}^2 &= (r_{00}^2 + r_{10}^2 + r_{01}^2 + r_{11}^2)\,e^{2i\alpha} \\ &= e^{2i\alpha} \quad \Longrightarrow \quad (r_{00}^2 + r_{10}^2 + r_{01}^2 + r_{11}^2) = 1 \end{aligned}$$

which can be phrased this way:

$$\mathbf{A} \text{ unitary} \implies \sum_{\mu,\nu=0}^{1} \bar{a}_{\mu\nu}a_{\mu\nu} = 1$$

We might (with misguided optimism) expect to be able—in principle—to argue similarly to a similar conclusion when n is arbitrary. But as will emerge, the computational details rapidly become overwhelming: it becomes clear that to carry out such a program a powerful new idea will be required.

Schwinger basis in the case n = 3. The eigenvalues of

$$\mathsf{P} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

are $\{1, \omega, \omega^2\}$ with $\omega = e^{i\frac{2\pi}{3}}$, so

$$\mathsf{Q} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}$$

and the Schwinger matrices $\mathsf{U}_{\mu\nu} = \mathsf{P}^\mu \mathsf{Q}^\nu : \mu, \nu \in \{0, 1, 2\}$ become

$$\begin{aligned}
\mathsf{U}_{00} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ \mathsf{U}_{01} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}, \ \mathsf{U}_{02} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix} \\
\mathsf{U}_{10} &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \ \mathsf{U}_{11} = \begin{pmatrix} 0 & \omega & 0 \\ 0 & 0 & \omega^2 \\ 1 & 0 & 0 \end{pmatrix}, \ \mathsf{U}_{12} = \begin{pmatrix} 0 & \omega^2 & 0 \\ 0 & 0 & \omega \\ 1 & 0 & 0 \end{pmatrix} \\
\mathsf{U}_{20} &= \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \ \mathsf{U}_{21} = \begin{pmatrix} 0 & 0 & \omega^2 \\ 1 & 0 & 0 \\ 0 & \omega & 0 \end{pmatrix}, \ \mathsf{U}_{22} = \begin{pmatrix} 0 & 0 & \omega \\ 1 & 0 & 0 \\ 0 & \omega^2 & 0 \end{pmatrix}
\end{aligned}$$

To say the same thing another way,

$$\begin{aligned}
\mathsf{A} &= \sum_{\mu,\nu=0}^{2} a_{\mu\nu} \mathsf{U}_{\mu\nu} \\
&= \begin{pmatrix} (a_{00} + a_{01} + a_{02}) & (a_{10} + \omega a_{11} + \omega^2 a_{12}) & (a_{20} + \omega^2 a_{21} + \omega a_{22}) \\ (a_{20} + a_{21} + a_{22}) & (a_{00} + \omega a_{01} + \omega^2 a_{02}) & (a_{10} + \omega^2 a_{11} + \omega a_{12}) \\ (a_{10} + a_{11} + a_{12}) & (a_{20} + \omega a_{21} + \omega^2 a_{22}) & (a_{00} + \omega^2 a_{01} + \omega a_{02}) \end{pmatrix}
\end{aligned}$$

Again, unitarity, tracelessness and trace-wise orthonormality—in the sense

$$\tfrac{1}{3}\text{tr}(\mathsf{U}^+_{\mu\nu} \mathsf{U}_{\rho\sigma}) = \delta_{\mu\rho}\delta_{\nu\sigma}$$

—are either obvious or readily verified. But now *all* of the $\mathsf{U}_{\mu\nu}$-matrices, with the sole exception of U_{00}, are non-hermitian:

$$\begin{aligned}
\mathsf{U}^+_{00} &= \mathsf{U}_{00} & \mathsf{U}^+_{01} &= \mathsf{U}_{02} & \mathsf{U}^+_{02} &= \mathsf{U}_{01} \\
\mathsf{U}^+_{10} &= \mathsf{U}_{20} & \mathsf{U}^+_{11} &= \omega^2 \mathsf{U}_{22} & \mathsf{U}^+_{12} &= \omega \mathsf{U}_{21} \\
\mathsf{U}^+_{20} &= \mathsf{U}_{10} & \mathsf{U}^+_{21} &= \omega \mathsf{U}_{12} & \mathsf{U}^+_{22} &= \omega^2 \mathsf{U}_{11}
\end{aligned}$$

so (noting that $\bar{\omega} = \omega^2$) we have

$$\mathsf{A}^+ = \begin{pmatrix} (\bar{a}_{00} + \bar{a}_{01} + \bar{a}_{02}) & (\bar{a}_{20} + \bar{a}_{21} + \bar{a}_{22}) & (\bar{a}_{10} + \bar{a}_{11} + \bar{a}_{12}) \\ (\bar{a}_{10} + \omega^2\bar{a}_{11} + \omega\bar{a}_{12}) & (\bar{a}_{00} + \omega^2\bar{a}_{01} + \omega\bar{a}_{02}) & (\bar{a}_{20} + \omega^2\bar{a}_{21} + \omega\bar{a}_{22}) \\ (\bar{a}_{20} + \omega\bar{a}_{21} + \omega^2\bar{a}_{22}) & (\bar{a}_{10} + \omega\bar{a}_{11} + \omega^2\bar{a}_{12}) & (\bar{a}_{00} + \omega\bar{a}_{01} + \omega^2\bar{a}_{02}) \end{pmatrix}$$

Using Schwinger's product rule to evaluate (with *Mathematica*'s assistance) the coefficients in

$$\mathsf{A}\,\mathsf{B} \equiv \mathsf{C} = \sum_{\kappa,\lambda=0}^{2} c_{\kappa\lambda}\,\mathsf{U}_{\kappa\lambda}$$

we are led quickly to results which, however, it seems pointless to spell out: each $c_{\kappa\lambda}$ is a sum a nine bilinear terms, with occasional terms decorated with powers of ω. For example, we find

$$\begin{aligned} c_{00} &= a_{00}b_{00} + a_{02}b_{01} + a_{01}b_{02} + a_{20}b_{10} + \omega a_{22}b_{11} \\ &\qquad + \omega^2 a_{21}b_{12} + a_{10}b_{20} + \omega^2 a_{12}b_{21} + \omega a_{11}b_{22} \\ c_{01} &= a_{01}b_{00} + a_{00}b_{01} + a_{02}b_{02} + \omega^2 a_{21}b_{10} + a_{20}b_{11} \\ &\qquad + \omega a_{22}b_{12} + \omega a_{11}b_{20} + a_{10}b_{21} + \omega^2 a_{12}b_{22} \\ &\;\;\vdots \end{aligned}$$

But when I attempted to solve the inversion problem by our former method —when I asked *Mathematica* to solve the symbolic system of nine equations in nine variables $b_{\rho\sigma}$ that results from setting $\{c_{00} = 1, c_{01} = c_{02} = \cdots = c_{22} = 0\}$ —my computer balked (ran out of memory and shut down). So I attempted to attack the inversion problem by another, more circumspect method:

Many years ago I established, and have over the years often made use of the fact,[18] that if $T_k = \mathrm{tr}\,\mathbb{A}^k$,

$$\begin{aligned} Q_0 &= 1 \\ Q_1 &= T_1 \\ Q_2 &= T_1^2 - T_2 \\ Q_3 &= T_1^3 - 3T_1T_2 + 2T_3 \\ &\;\;\vdots \end{aligned}$$

and (in the 3-dimensional case)

$$\begin{aligned} p_0 &= +\tfrac{1}{3!}Q_3 \\ p_1 &= -\tfrac{1}{2!}Q_2 \\ p_2 &= +\tfrac{1}{1!}Q_1 \\ p_3 &= -1 \end{aligned}$$

[18] See "A mathematical note: Algorithm for the efficient evaluation of the trace of the inverse of a matrix" (1996).

then the denominator/numerator in

$$\mathbb{A}^{-1} = \frac{\text{transposed matrix of cofactors}}{\det \mathbb{A}}$$

admit of trace-wise description as follows:

$$\det \mathbb{A} = \tfrac{1}{6}(T_1^3 - 3T_1T_2 + 2T_3)$$

and

$$\begin{aligned}\text{transposed matrix of cofactors} &= -(p_1\mathbb{I} + p_2\,\mathbb{A} - \mathbb{A}^2)\\ &= \mathbb{A}^2 - T_1\mathbb{A} + \tfrac{1}{2}(T_1^2 - T_2)\,\mathbb{I}\\ &= \Big[\mathbb{A} - \tfrac{1}{2}\big(T_1 + \sqrt{2T_2 - T_1T_1}\,\big)\mathbb{I}\Big]\Big[\mathbb{A} - \tfrac{1}{2}\big(T_1 - \sqrt{2T_2 - T_1T_1}\,\big)\mathbb{I}\Big]\end{aligned}$$

These results[19]—despite their unfamiliar appearance—are found to pass numerical tests. Notice that we have once again managed to *factor* the numerator (the transposed matrix of cofactors), but that the factors—which are, in an obvious formal sense, "conjugates" of one another—now contain surds, which in the circumstances presented by the Pauli and Dirac algebras they did not.[20]

The preceding solution of the inversion problem imposes no limitation on the *manner* in which we elect to present the 3×3 matrix in question; we might, in paticular, elect to write either of the following

$$\mathbf{A} = \begin{pmatrix} A_{11} & A_{12} & A_{13}\\ A_{21} & A_{22} & A_{23}\\ A_{31} & A_{32} & A_{33}\end{pmatrix} = \sum_{\mu,\nu=0}^{2} a_{\mu\nu}\,\mathbf{U}_{\mu\nu}$$

But in either case

$$\mathbf{A}\cdot\det\mathbf{A} = \big[\mathbf{A} - \tfrac{1}{2}\big(T_1 + \sqrt{2T_2 - T_1T_1}\,\big)\mathbf{I}\big]\big[\mathbf{A} - \tfrac{1}{2}\big(T_1 - \sqrt{2T_2 - T_1T_1}\,\big)\mathbf{I}\big]$$

is an <u>uninformative quadratic mess</u> that provides no insight into the conditions that *unitarity*

$$\mathbf{A}\cdot\det\mathbf{A} = \mathbf{A}^{+}$$

imposes on the elements (or Schwinger coordinates) of $\mathbf{A}$.

[19] In the 2-dimensional case we find

$$\begin{gathered}\det\mathbb{A} = \tfrac{1}{2}(T_1^2 - T_2)\\ \text{transposed matrix of cofactors} = T_1\,\mathbb{I} - \mathbb{A}\end{gathered}$$

When the matrix is presented in Schwinger-expanded form we are led efficiently back again to the inversion formula obtained in the preceding section.

[20] When the dimension $n \geq 5$ we might expect—by the "insolubility of the quintic"—the analogs of those surds to be generally (in the absence of special circumstances) *undescribable*!

It may be of interest to note that if we possessed a set $\{\mathsf{R}_0, \mathsf{R}_1, \mathsf{R}_2, \ldots, \mathsf{R}_8\}$ of 3×3 matrices that are (like Pauli and Dirac matrices) trace-wise orthonormal in the "*un*adjointed" sense

$$\tfrac{1}{3}\mathrm{tr}(\mathsf{R}_i\,\mathsf{R}_j) = \delta_{ij}$$

and if $\mathsf{R}_0 = \mathsf{I}$ (which renders all the other R-matrices traceless) then it becomes possible to write

$$\begin{aligned} \mathsf{A} &= \sum_{k=0}^{8} a_k\,\mathsf{R}_k \qquad \text{with} \qquad a_k = \tfrac{1}{3}\mathrm{tr}(\mathsf{R}_k\,\mathsf{A}) \\ &\Downarrow \\ \mathrm{tr}\,\mathsf{A} &= 3a_0 \\ \mathrm{tr}\,\mathsf{A}^2 &= 3(a_0^2 + a_1^2 + \cdots + a_8^2) \end{aligned}$$

which serve quite effectively to tidy-up the "quadratic mess," but still do not place us in position to formulate a sharp unitarity condition. R-matrices with the stipulated properties can always be fabricated by means of a "trace-wise Gram-Schmidt orthogonalization procedure," but that is hardly worth the effort, for we have by this point exhausted the merits of the approach: in higher dimensions (or to evaluate the determinant even in this 3-dimensional case) one has necessarily to confront the complications latent in the composition law

$$\mathsf{R}_i\,\mathsf{R}_j = \sum_{ij} c_{ijk}\,\mathsf{R}_k$$

Numerical evidence supports the claim that

$$\mathsf{A} \text{ unitary} \implies \sum_{\mu,\nu=0}^{2} \bar{a}_{\mu\nu}a_{\mu\nu} = 1$$

In the 2-dimensional case we found it fairly easy to proceed from a solution of the inversion problem to a unitarity criterion that permitted one to supply an algebraic proof of the validity of such a claim. I have belabored the 3-dimensional theory in that hope that I might enjoy similar success, and thus be guided toward a proof that works for arbitrary dimension. But that goal seems now to remain as unappoachably distant as ever.[21] I have developed an appreciation for why it was that Weyl/Schwinger/Werner were content to ignore the issue.

Oppenheim's contribution. It was, as I indicated at the outset, a remark by Jonathan Oppenheim and his co-author that put me onto this subject. In

[21] I am reminded that it was Hamilton's prolonged effort to solve "the division problem" that led him at length to the invention of quaternions.

their text, Oppenheim & Reznik appear to attribute the unitary basis idea to Reinhard Werner[5,6], but in the Appendix of their paper[3] they sketch a method for constructing unitary bases that differs markedly from Werner's, in connection with which they mention a half-century old paper by Schwinger.[7] Schwinger's paper is diffuse, and I found I had to work to separate the material relating specifically to basis construction from other ideas explored in that somewhat opaque paper. I came at length to the realization that Oppenheim and Reznik had done similar work, and that their unitary basis construction procedure is borrowed *directly* from Schwinger. What I learned from Oppenheim is how fundamentally *simple* the unitary basis idea is—a perception not easy to garner from Schwinger, and certainly not from Werner.

Werner's unitary bases. Werner restricts his attention to unitary matrices of the form

$$\mathsf{U} = (\text{permutation matrix}) \cdot (\text{unitary diagonal})$$

Pauli matrices are of this form, so are Dirac matrices, and so are the matrices contemplated by Schwinger.

Permutation matrices have a 1 in each row/column, with all other elements zero. Permutation matrices P are in all cases inverted by transposition: all such matrices are therefore rotation matrices (real unitary matrices), proper or improper according as the permutation they accomplish is even or odd.

Diagonal matrices Q are unitary if and only the elements on the principle diagonal are of the form $e^{i\alpha}$.

Schwinger's procedure is "rigid" in the sense that he assigns a specific value to P (assumes P to describe a one-step cyclic advance) and distributes the eigenvalues $\{\omega^0, \omega^1, \ldots, \omega^{n-1}\}$ of P along the principal diagonal of Q. It does acquire some flexibility from the observation that where Schwinger writes

$$\mathsf{U}_{\mu\nu} = \mathsf{P}^{\mu}\mathsf{Q}^{\nu}$$

the role assigned to P could be reassigned to any power p of P that does not divide n, and the role of Q reassigned to any power q of Q that is similarly constrained, but those adjustments serve merely to shuffle the labels worn by the $\mathsf{U}_{\mu\nu}$ matrices. And of course, when all is said and done one could construct alternative bases by means of unitary similarity transformations

$$\mathsf{U}_{\mu\nu} \longrightarrow \mathsf{U}_{\mu\nu}^{\text{new}} = \mathsf{S}^{-1}\mathsf{U}_{\mu\nu}\mathsf{S}$$

that preserve unitarity and all trace properties, but destroy the shifted diagonal structure of the original matrices.

Werner matrices are defined

$$\mathsf{W}_{ij} \equiv \|W_{ij,pq}\| = \|H_{ip}\ \text{KroneckerDelta}[q, L_{jp}]\|$$

where the H_{ip} are elements of an $n \times n$ **Hadamard matrix** $\mathbb{H}$ and the L_{jp} are elements of an $n \times n$ **Latin square** $\mathbb{L}$. The Werner construction derives its relatively greater flexibility from the circumstance that for $n \geq 4$ the matrices $\mathbb{H}$ and (especially) $\mathbb{L}$ can be selected from a rapidly expanding set of possibilities.

Latin squares[22] are square arrangements of symbols (call them 1, 2, . . . , n) in which each symbol appears exactly once in each row and column:

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 4 & 1 & 2 \\ 4 & 3 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \\ 3 & 1 & 4 & 2 \\ 4 & 3 & 2 & 1 \end{pmatrix}$$

Some (such as those shown above) can be read as group tables, but others cannot; the following is the smallest example of a Latin square that can be interpreted to refer not to a group but to a "quasi-group" (non-associative group, or "loop"):

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 1 & 5 & 3 \\ 3 & 5 & 4 & 2 & 1 \\ 4 & 1 & 5 & 3 & 2 \\ 5 & 3 & 2 & 1 & 4 \end{pmatrix}$$

But the latter serves Werner's purpose just as well as the others. The literature describes the criteria with respect to which Latin squares of the same dimension become "equivalent/inequivalent." The number of inequivalent Latin squares is a very rapidly increasing function of dimension: at $n = 2^3$ it has become 283657, and by n = 10 it has reportedly grown to $34817397894749939 \approx 3.48 \times 10^{16}$, which affords Werner plenty of room in which to wiggle!

The theory of Hadamard matrices[23] originates in a paper[24] by J. J. Sylvester, and acquired its name from a paper published twenty-six years later by Jacques Hadamard. Hadamard matrices are square matrices with all elements equal to ± 1 and with the further property that all rows/columns are orthogonal, which entails

$$\mathbb{H}\,\mathbb{H}^{\mathsf{T}} = n\,\mathbb{I}$$

In the simplest instance one has

$$\mathbb{H}_2 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

Sylvester himself contemplated matrices of progressively higher order

$$\mathbb{H}_4 = \mathbb{H}_2 \otimes \mathbb{H}_2 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}, \ \mathbb{H}_8 = \mathbb{H}_2 \otimes \mathbb{H}_4\,, \ \text{etc.}$$

---

[22] Consult http://en.wikipedia.org/wiki/Latin_square.

[23] See http://en.wikipedia.org/wiki/Hadamard_matrix.

[24] "Thoughts on inverse orthogonal matrices, simultaneous sign successions, and tessellated pavements in two or more colors, with applications to Newton's rule, ornamental tile-work, and the theory of numbers," Phil. Mag. **34** 461–475, (1867).

Such matrices have dimension $\{2, 4, 8, 16, 32, 64, \ldots, 2^\nu\}$. The still-unproven *Hadamard conjecture* asserts that real Hadamard matrices exist in all dimensions that are multiples of 4, which would fill in these gaps in Sylvester's list: $\{12, -, 20, 24, 28, -, 36, 40, 44, 48, 52, 56, 60, -, \ldots\}$. As of 2008, the least value of n for which Hadamard's conjecture has not been confirmed is $n = 688 = 4 \times 172$, and there were a total of thirteen such cases with $n < 2000$. The real Hadamard matrices are (given the natural interpretation of "equivalence" supplied by the literature) unique through $n = 2, 4, 6, 12$, but 5 inequivalent Hadamard matrices exist for $n = 16$, and millions are known for $n \geq 32$. This again provides Werner with plenty or room to wiggle, at least in dimensions that are multiples of four.

Complex Hadamard matrices—which satisfy the complexified condition

$$\mathbb{H}\,\mathbb{H}^{+} = n\,\mathbb{I}$$

—exist in *all* dimensions, including those that are *not* multiples of four. The most important class of such matrices are those of "Butson type,"[25] which in n-dimensions possess the "Fourier structure"

$$\mathbb{F}_n = \|F_{n,jk}\| \quad \text{with} \quad F_{n,ij} = \omega^{(j-1)(k-1)}$$

where $\omega = e^{i\,2\pi/n}$ and $i, j \in \{1, 2, \ldots, n\}$. Low-dimensional examples look like this:

$$\mathbb{F}_2 = \begin{pmatrix} 1 & 1 \\ 1 & \omega \end{pmatrix} \qquad\qquad \text{with} \quad \omega = e^{i\,2\pi/2} = -1$$

$$\mathbb{F}_3 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix} \qquad\qquad \text{with} \quad \omega = e^{i\,2\pi/3}$$

$$\mathbb{F}_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & \omega^3 \\ 1 & \omega^2 & 1 & \omega^2 \\ 1 & \omega^3 & \omega^2 & \omega \end{pmatrix} \quad \text{with} \quad \omega = e^{i\,2\pi/4} = i$$

I will not attempt to demonstrate the unitarity, tracelessness or trace-wise orthonormality of Werner's matrices $\mathbf{W}_{ij}$—the arguments are a bit intricate; the Schwinger matrices $\mathbf{U}_{\mu\nu}$ serve all practical purposes, and for those such demonstrations are already in hand—but will be content to look to specific examples of Werner bases in some low-dimensional cases.

The Werner basis in the case n = 2. In this case the selection of $\mathbb{H}$ and $\mathbb{L}$ is not optional. One has

$$\mathbb{H} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad \text{and} \quad \mathbb{L} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$

[25] See http://en.wikipedia.org/wiki/Butson-type_Hadamard_matrices. The original reference is A. T. Butson, "Generalized Hadamard matrices," Proc. Amer. Math. Soc. **13**, 894-898 (1962).

Entrusting the computational labor to *Mathematica*, Werner's construction supplies

$$\mathsf{W}_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ \mathsf{W}_{12} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \mathsf{W}_{21} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ \mathsf{W}_{11} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

which are identical (apart from labeling and order) to the matrices supplied in the 2-dimensional case by Schwinger's construction.

The Werner basis in the case n = 3. The selection of $\mathbb{H}$ and $\mathbb{L}$ is again not optional. One has

$$\mathbb{H} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix} \quad \text{and} \quad \mathbb{L} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{pmatrix}$$

giving

$$\begin{aligned}
\mathsf{W}_{11} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & \mathsf{W}_{12} &= \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, & \mathsf{W}_{13} &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \\
\mathsf{W}_{21} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}, & \mathsf{W}_{22} &= \begin{pmatrix} 0 & 0 & \omega^2 \\ 1 & 0 & 0 \\ 0 & \omega & 0 \end{pmatrix}, & \mathsf{W}_{23} &= \begin{pmatrix} 0 & \omega & 0 \\ 0 & 0 & \omega^2 \\ 1 & 0 & 0 \end{pmatrix} \\
\mathsf{W}_{31} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix}, & \mathsf{W}_{32} &= \begin{pmatrix} 0 & 0 & \omega \\ 1 & 0 & 0 \\ 0 & \omega^2 & 0 \end{pmatrix}, & \mathsf{W}_{33} &= \begin{pmatrix} 0 & \omega^2 & 0 \\ 0 & 0 & \omega \\ 1 & 0 & 0 \end{pmatrix}
\end{aligned}$$

where now $\omega = e^{i\,2\pi/3}$. This is again identical to the population of unitary matrices produced by Schwinger's construction.

Werner bases in the case n = 4. This is case of least dimension in which we confront inequivalent (but equally viable) options: we can select either of two possible Hadamard matrices (one real, the other complex), and either of two possible Latin squares. I look initially to the Werner matrices that result from selecting

$$\mathbb{H} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \quad \text{and} \quad \mathbb{L} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 4 & 1 & 2 \\ 4 & 3 & 2 & 1 \end{pmatrix}$$

because the reality of $\mathbb{H}_{\text{Sylvester}}$ has this interesting consequence: it gives rise to *real-valued* unitary matrices, which is to say: orthonormal *rotation* matrices.

Mathematica supplies

$$
\begin{array}{cccc}
\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}, &
\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}, &
\begin{pmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0\end{pmatrix}, &
\begin{pmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{pmatrix}, \\[2mm]
\begin{pmatrix}1&0&0&0\\0&\omega&0&0\\0&0&1&0\\0&0&0&\omega\end{pmatrix}, &
\begin{pmatrix}0&\omega&0&0\\1&0&0&0\\0&0&0&\omega\\0&0&1&0\end{pmatrix}, &
\begin{pmatrix}0&0&1&0\\0&0&0&\omega\\1&0&0&0\\0&\omega&0&0\end{pmatrix}, &
\begin{pmatrix}0&0&0&\omega\\0&0&1&0\\0&\omega&0&0\\1&0&0&0\end{pmatrix}, \\[2mm]
\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&\omega&0\\0&0&0&\omega\end{pmatrix}, &
\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&\omega\\0&0&\omega&0\end{pmatrix}, &
\begin{pmatrix}0&0&\omega&0\\0&0&0&\omega\\1&0&0&0\\0&1&0&0\end{pmatrix}, &
\begin{pmatrix}0&0&0&\omega\\0&0&\omega&0\\0&1&0&0\\1&0&0&0\end{pmatrix}, \\[2mm]
\begin{pmatrix}1&0&0&0\\0&\omega&0&0\\0&0&\omega&0\\0&0&0&1\end{pmatrix}, &
\begin{pmatrix}0&\omega&0&0\\1&0&0&0\\0&0&0&1\\0&0&\omega&0\end{pmatrix}, &
\begin{pmatrix}0&0&\omega&0\\0&0&0&1\\1&0&0&0\\0&\omega&0&0\end{pmatrix}, &
\begin{pmatrix}0&0&0&1\\0&0&\omega&0\\0&\omega&0&0\\1&0&0&0\end{pmatrix}
\end{array}
$$

where $\omega = -1$. W_{ij} is found in the i^{th} row and j^{th} column of that display. If, on the other hand, we set $\mathbb{H} = \mathbb{F}_4$ (but retain the same $\mathbb{L}$ as before) we obtain

$$
\begin{array}{cccc}
\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix} &
\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix} &
\begin{pmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0\end{pmatrix} &
\begin{pmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{pmatrix} \\[2mm]
\begin{pmatrix}1&0&0&0\\0&\omega&0&0\\0&0&\omega^2&0\\0&0&0&\omega^3\end{pmatrix} &
\begin{pmatrix}0&\omega&0&0\\1&0&0&0\\0&0&0&\omega^3\\0&0&\omega^2&0\end{pmatrix} &
\begin{pmatrix}0&0&\omega^2&0\\0&0&0&\omega^3\\1&0&0&0\\0&\omega&0&0\end{pmatrix} &
\begin{pmatrix}0&0&0&\omega^3\\0&0&\omega^2&0\\0&\omega&0&0\\1&0&0&0\end{pmatrix} \\[2mm]
\begin{pmatrix}1&0&0&0\\0&\omega^2&0&0\\0&0&1&0\\0&0&0&\omega^2\end{pmatrix} &
\begin{pmatrix}0&\omega^2&0&0\\1&0&0&0\\0&0&0&\omega^2\\0&0&1&0\end{pmatrix} &
\begin{pmatrix}0&0&1&0\\0&0&0&\omega^2\\1&0&0&0\\0&\omega^2&0&0\end{pmatrix} &
\begin{pmatrix}0&0&0&\omega^2\\0&0&1&0\\0&\omega^2&0&0\\1&0&0&0\end{pmatrix} \\[2mm]
\begin{pmatrix}1&0&0&0\\0&\omega^3&0&0\\0&0&\omega^2&0\\0&0&0&\omega\end{pmatrix} &
\begin{pmatrix}0&\omega^3&0&0\\1&0&0&0\\0&0&0&\omega\\0&0&\omega^2&0\end{pmatrix} &
\begin{pmatrix}0&0&\omega^2&0\\0&0&0&\omega\\1&0&0&0\\0&\omega^3&0&0\end{pmatrix} &
\begin{pmatrix}0&0&0&\omega\\0&0&\omega^2&0\\0&\omega^3&0&0\\1&0&0&0\end{pmatrix}
\end{array}
$$

where now $\omega = i$, $\omega^2 = -1$ and $\omega^3 = -i$. Note that the matrices in the first row—unchanged from before—are real-valued, but so also are the matrices in the third row. More interesting is the observation that the preceding list is distinct from the list of Dirac matrices, which can be extablished by the following simple consideration: the Dirac list contains four members—namely

$\{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}$—that square to I and anticommute. The Werner list (we are informed by *Mathematica*) contains six members that square to the identity

$$\mathsf{W}_{11}\mathsf{W}_{11} = \mathsf{W}_{12}\mathsf{W}_{12} = \mathsf{W}_{13}\mathsf{W}_{13} = \mathsf{W}_{14}\mathsf{W}_{14} = \mathsf{W}_{31}\mathsf{W}_{31} = \mathsf{W}_{33}\mathsf{W}_{33} = \mathsf{I}$$

from which it is possible to select two distinct anticommutative triples (namely $\{\mathsf{W}_{12}, \mathsf{W}_{31}, \mathsf{W}_{33}\}$ and $\{\mathsf{W}_{14}, \mathsf{W}_{31}, \mathsf{W}_{33}\}$) but it is not possible to select an anticommutative quartet.[26] The Werner list is distinct also from Schwinger's list of 4×4 unitaries, for—though there are some coincidences—the latter contains six members (three each) of designs

$$\begin{pmatrix} 0 & 0 & 0 & \bullet \\ \bullet & 0 & 0 & 0 \\ 0 & \bullet & 0 & 0 \\ 0 & 0 & \bullet & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & \bullet & 0 & 0 \\ 0 & 0 & \bullet & 0 \\ 0 & 0 & 0 & \bullet \\ \bullet & 0 & 0 & 0 \end{pmatrix}$$

that are absent from Werner's list. Nor does selection of the other one of the two available Latin squares bring the Werner and Schwinger lists into agreement.

It is of interest to note finally that the dimensions $n = 2^\nu$ of real Hadamard matrices of Sylvester's design—whence of real (and therefore rotational) Werner bases—are precisely the dimensions of the state spaces that support the quantum dynamics of ν qubits. Real Warner bases are, however, not that rare: they (by the Hadamard conjecture) can be constructed whenever the dimension is a multiple of four.

ADDENDUM: Randomly constructed unitary bases. The eigenvalues of a unitary matrix U have the form $e^{i\alpha_k}$, where the real-valued α_k are the eigenvalues of the of the hermitian eigenvalues of the generator of U. We have

[26] Had the situation turned out otherwise I would have searched for a unitary matrix S that by similarity transformation sends Werner $\longrightarrow$ Dirac, since all representations of the Dirac algebra are known to be similarity-equivalent.